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Abstract: Fitting's indexed nested sequents can be used to give deductive systems to modal logics which cannot be captured by pure nested sequents. In this paper we show how the standard cut-elimination procedure for nested sequents can be extended to indexed nested sequents, and we discuss how indexed nested sequents can be used for intuitionistic modal logics.

Key-words: Indexed nested sequents, intuitionistic modal logics, cut elimination

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Sur la théorie de la démonstration en calcul des séquents emboités indexés pour les logiques modales classiques et intuitionnistes

Résumé : Le calcul des séquents emboîtés indexés introduit par Fitting permet de construire des systèmes de preuves pour certaines logiques modales n'ayant pas de système en calcul des séquents emboîtés purs. Dans cet article, nous exposons une preuve d'élimination des coupures pour le calcul des séquents indexés en adaptant la preuve pour le calcul pur, et nous étudions comment traîter de logiques modales intuitionnistes en calcul des séquents emboîtés indexés.

Mots-clés : sequents emboîtés indexés, logique modale intuitionniste, élimination des coupures

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1 Introduction

Modal logics were originally defined in terms of axioms in a Hilbert system, and later in terms of their semantics in relational structures. Structural proof theory for modal logics, however, was considered a difficult topic as traditional (Gentzen) sequents did not provide fully satisfactory (i.e. analytic and modular) proof systems even for some common modal logics. Nonetheless, the proof theory of modal logics has received more attention in the last decade, and some extensions of traditional sequents were successfully proposed to handle modalities. Two approaches can be distinguished: systems that incorporate relational semantics in the formalism itself like labelled sequents systems, and systems that use syntactical devices to handle the modalities like nested sequents (aka. tree-hypersequents) or display calculus.

Labelled sequents (e.g., [Rus96,Vig00,Neg05]) are a versatile framework that can give deductive systems for a large class of modal logics using sequents that explicitly refer to the relational semantics: formulas are labelled with states and relational atoms describe the accessibility relation.

Nested sequents are an extension of ordinary sequents to a structure of tree, first introduced by Kashima [Kas94], and then independently rediscovered by Brünnler [Brü09] and Poggiolesi [Pog09]. They can be translated into a subclass of labelled sequents called in [GR12] labelled tree sequents, if the relational structure is made explicit. However, compared to labelled deductive systems, the tree structure restricts the expressivity of nested sequents. In particular, it seems that nested sequents cannot give cut-free deductive systems for logics obeying the Scott-Lemmon axioms, which correspond to a "confluence" condition on the relational structure [LS77].

Fitting recently introduced *indexed nested sequents* [Fit15], an extension of nested sequents which goes beyond the tree structure to give a cut-free system for the classical modal logic K extended with an arbitrary set of Scott-Lemmon axioms. In some sense indexed nested sequents are more similar to labelled systems than pure nested sequents—in fact, the translation between nested sequents and labelled tree sequents mentioned above is naturally extended in [Ram16] to a translation between indexed nested sequents and labelled tree sequents with equality, where some nodes of the underlying tree can be identified. On the other

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hand, indexed nested sequents can still be seen as an internal formalism because it does not need to make use of the relational semantics in the formulas syntax. In this way, one could argue that indexed nested sequents bring together the best of both worlds, nested sequents and labelled sequents systems.

In this paper we investigate some proof-theoretical properties of indexed nested sequents. We are interested in transferring results from the nested sequents meta-theory to the indexed setting. The first and foremost one is the cut-elimination theorem, that shows that the proof system is complete without the cut-rule. As Fitting's original system does not use a cut rule, this result is actually entailed by his (semantical) completeness theorem. There exists also a (indirect) syntactic proof of cut-elimination via the translation to labelled tree sequents with equality reusing the existing cut-elimination for labelled systems [Ram16]. However, from the point of structural proof theory, it is advantageous that a syntactic proof of cut-elimination can be carried out within the considered proof formalism. For this reason we give in this paper an internal proof of cut-elimination for indexed nested sequents in the traditional way of a list of rewriting cases to directly show the admissibility of the cut-rule. We achieve our result by making some subtle but crucial adjustments to the standard cut-elimination proof for pure nested sequents.

One of the main advantages is that this proof can be exported to the intuitionistic framework with basically no effort. Indeed, our second interest was to extend Fitting's system, that only considered classical modal logics, to the intuitionistic framework by using the techniques that had already been successfully used for ordinary nested sequents [GS10,Str13,MS14]. In the second part of the paper, we discuss the extension of our framework to the intuitionistic setting. More precisely, we present the cut-free indexed nested sequents systems in a uniform manner for classical and intuitionistic modal logic. The deductive systems are almost identical, the main difference being that an intuitionistic sequent has only one "output" formula, in the same way as in ordinary sequent calculus an intuitionistic sequent has only one formula on the right.

As there is no straightforward definition of the extension of intuitionistic modal logic with Scott-Lemmon axioms, the indexed nested sequents system can be seen as one way to define it. This point is examined in the last section with a discussion on the various alternatives that exist in the literature and how they relate to the proposed system.

2 Indexed nested sequents and the Scott-Lemmon axioms

We start by working with formulas in negation normal form, from the following grammar, which extends the language of propositional classical logic with the two modalities $\Box$ and $\Diamond$

$$A ::= a \mid \bar{a} \mid A \wedge A \mid A \vee A \mid \Box A \mid \Diamond A \tag{1}$$

where a is taken from a countable set of propositional atoms, $\bar{a}$ is its negation, and $\bar{\bar{a}}$ is equivalent to a. For every formula A, its negation $\bar{A}$, is defined as usual via the De Morgan laws. For now, we use $A \supset B$ as abbreviation for $\bar{A} \vee B$.

Classical modal logic K is obtained from classical propositional logic by adding the axiom k: $\Box(A \supset B) \supset (\Box A \supset \Box B)$ and the *necessitation rule* that allows to derive the formula $\Box A$ from any theorem A.

Stronger modal logics can be obtained by adding to K other axioms. In this paper we are interested specifically in the family of *Scott-Lemmon axioms* of the form

$$\mathsf{g_{k,l,m,n}} : \Diamond^k \Box^l A \supset \Box^m \Diamond^n A \tag{2}$$

for a tuple $\langle k, l, m, n \rangle$ of natural numbers, where $\Box^m$ stands for m boxes and $\Diamond^n$ for n diamonds.

Fitting [Fit15] introduced indexed nested sequents exactly to provide a structural proof system for classical modal logic K, that could be extended with rules for the Scott-Lemmon axioms.

A *(pure) nested sequent* is a multiset of formulas and *boxed sequents*, according to the following grammar $\Gamma ::= \emptyset \mid A, \Gamma \mid [\Gamma], \Gamma$ where A is a modal formula.We understand such a nested sequent through its interpretation as a modal formula, written $fm(\cdot)$, given inductively by $fm(\emptyset) = \bot$; $fm(A, \Gamma) = A \vee fm(\Gamma)$; and $fm([\Gamma_1], \Gamma_2) = \Box fm(\Gamma_1) \vee fm(\Gamma_2)$. A nested sequent can therefore be seen as a tree of ordinary *one-sided* sequents, with each node representing the scope of a modal $\Box$. It therefore is of the general form

$$A_1, \ldots, A_k, [\Gamma_1], \ldots, [\Gamma_n] \tag{3}$$

An *indexed nested sequent*, as defined in [Fit15], is a nested sequent where each sequent node (either the root or any interior node) carries an *index*, denoted by lowercase letters like $u, v, w, x, \ldots$, and taken from a countable set (e.g., for simplicity, the set of natural numbers), so we write an indexed sequent by extending (3) in the following way

$$A_1, \ldots, A_k, [^{w_1} \Gamma_1], \ldots, [^{w_n} \Gamma_n] \tag{4}$$

where $\Gamma_1, \ldots, \Gamma_n$ are now indexed sequents, and where the index of the root is not explicitly shown (e.g., we can assume that it is 0). For an indexed nested sequent Σ, we write I_Σ to denote the set of indexes occurring in Σ.

Intuitively, once indexed, nested sequents are no longer trees, but any kind of *rooted* directed graphs[1] (possibly further constrained by some conditions on the indexing), by identifying nodes carrying the same index. Indeed the structure of a rooted directed graph is equivalent to that of a tree where certain nodes are identified.

In nested sequent calculi, a rule can be applied at any depth in the structure, that is, inside a certain nested sequent context. We write $\Gamma^{i_1}\{\ \} \cdots {}^{i_n}\{\ \}$ for an n-ary context (i.e. one with n occurrences of the $\{\ \}$) where $i_1, \ldots, i_n$ are the

[1] A *rooted* graph is a graph where one node is distinguished as the root and every node is reachable from it, i.e., the whole graph can be obtained as the minimal upward closure of this root for the edge relation.

$$\mathsf{id}\,\frac{}{\Gamma\{a,\bar{a}\}} \quad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \quad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \quad \Diamond\,\frac{\Gamma\{\Diamond A,[^u A,\Delta]\}}{\Gamma\{\Diamond A,[^u\Delta]\}} \quad \Box\,\frac{\Gamma\{[^v A]\}}{\Gamma\{\Box A\}}\; v \text{ is fresh}$$

$$\mathsf{tp}\,\frac{\Gamma^{\,w}\{\emptyset\}^{\,w}\{A\}}{\Gamma^{\,w}\{A\}^{\,w}\{\emptyset\}} \qquad \mathsf{bc}_1\,\frac{\Gamma^{\,w}\{[^u\Delta]\}^{\,w}\{[^u\emptyset]\}}{\Gamma^{\,w}\{[^u\Delta]\}^{\,w}\{\emptyset\}} \qquad \mathsf{bc}_2\,\frac{\Gamma^{\,w}\{[^u\Delta^{\,w}\{[^u\emptyset]\}]\}}{\Gamma^{\,w}\{[^u\Delta^{\,w}\{\emptyset\}]\}}$$

Fig. 1. System iNK

$$\mathsf{g_{k,l,m,n}}\,\frac{\Gamma^{\,u_0}\{[^{u_1}\Delta_1,\ldots[^{u_k}\Delta_k,[^{v_1}\ldots[^{v_l}]\ldots]]\ldots],[^{w_1}\Sigma_1,\ldots[^{w_m}\Sigma_m,[^{x_1}\ldots[^{x_n}]\ldots]]\ldots]\}}{\Gamma^{\,u_0}\{[^{u_1}\Delta_1,\ldots[^{u_k}\Delta_k]\ldots],[^{w_1}\Sigma_1,\ldots[^{w_m}\Sigma_m]\ldots]\}}$$

Fig. 2. Inference rule $\mathsf{g_{k,l,m,n}}$ (where $l+n\neq 0$, and $c_l = d_n$)

indexes of the sequent nodes that contain the $\{\;\}$, in the order of their appearance in the sequent. A hole in a context can be replaced by a formula or sequent. More precisely, we write $\Gamma^{\,i_1}\{\Delta_1\}\cdots{}^{i_n}\{\Delta_n\}$ for the sequent that is obtained from $\Gamma^{\,i_1}\{\;\}\cdots{}^{i_n}\{\;\}$ by replacing the k-th hole by Δ_k, for each $k\in\{1,\ldots,n\}$ (if $\Delta_k=\emptyset$ it simply amounts to removing the $\{\;\}$). We might omit the index at the context-braces when this information is clear or not relevant.

Example 2.1 For example, $A,[^1 B,[^2 C,\{\;\}]],[^3 D,[^1\{\;\},A]],[^2 D,\{\;\}]$ is a ternary context that we can write as $\Gamma^{\,2}\{\;\}^{\,1}\{\;\}^{\,2}\{\;\}$. If we substitute the sequents $\Delta_1 = D,[^4 E]$; $\Delta_2 = F$; and $\Delta_3 = [^5 G]$ into its holes, we get: $\Gamma^{\,2}\{\Delta_1\}^{\,1}\{\Delta_2\}^{\,2}\{\Delta_3\} = A,[^1 B,[^2 C,D,[^4 E]]],[^3 D,[^1 F,A]],[^2 D,[^5 G]]$

In Figure 1, the classical system that we call iNK is an adaptation of the system described by Fitting in [Fit15] to our notations and to the one-sided setting. It can also be seen as Brünnler's system [Brü09] extended with indexes. From now on, a formal *proof* is defined as a derivation tree constructed according to the rules of the calculus iNK. By derivation tree, we mean a rooted tree in which: every leaf is labeled by an axiom rule, every node is labeled by a rule of the calculus iNK, every edge is labeled by a sequent.

What is different from the pure nested sequent system is the addition of the two structural rules tp and bc, called *teleportation* and *bracket-copy*, respectively, which are variants of the formula-contraction FC and the sequent-contraction SC of [Fit15]. Note that we need two versions of bc to take care of every possible context where the rule my be applied. They are needed to adapt the system to the indexed sequents, namely to maintain the intended semantics by allowing two brackets with the same index to be identified. Another peculiarity is that in the rules for $\Box$ we demand that the index of the new bracket in the premiss does not occur in the conclusion.

Finally, for a tuple $\langle k,l,m,n\rangle$ with $l+n\neq 0$, the rule $\mathsf{g_{k,l,m,n}}$ in Figure 2 is defined as in [Fit15]. It must satisfy that $v_1\ldots v_k$ and $x_1\ldots x_n$ are fresh indexes

$$\mathsf{g}_{\mathsf{k,0,m,0}}\ \frac{\sigma\Gamma^{u_0}\{[^{u_1}\Delta_1,\ldots[^{\sigma(u_k)}\Delta_k],\ldots],[^{w_1}\Sigma_1,\ldots[^{\sigma(w_m)}\Sigma_m],\ldots]\}}{\Gamma^{u_0}\{[^{u_1}\Delta_1,\ldots[^{u_k}\Delta_k],\ldots],[^{w_1}\Sigma_1,\ldots[^{w_m}\Sigma_m],\ldots]\}}$$

Fig. 3. Special case for $\mathsf{g}_{\mathsf{k,0,m,0}}$

$$\mathsf{cut}\ \frac{\Gamma\{A\}\quad \Gamma\{\bar{A}\}}{\Gamma\{\emptyset\}} \quad\Bigg|\quad \mathsf{w}\ \frac{\Gamma\{\emptyset\}}{\Gamma\{\Delta\}} \qquad \mathsf{c}\ \frac{\Gamma\{\Delta,\Delta\}}{\Gamma\{\Delta\}} \qquad \mathsf{nec}\ \frac{\Gamma}{[\Gamma]} \qquad \mathsf{isub}\ \frac{\Gamma}{\sigma\Gamma}$$

Fig. 4. Left: The one-sided cut-rule – **Right:** Additional structural rules

which are pairwise distinct, except for the *confluence condition*: we always have $v_l = x_n$. When one or more elements of the tuple $\langle k,l,m,n\rangle$ are equal to 0, there are special cases:

- if $k = 0$ (or $m = 0$) then u_1 to u_k (resp. w_1 to w_m) all collapse to u_0.
- if $l = 0$ then w_1 to w_l all collapse to u_k, and similarly, if $n = 0$ then x_1 to x_n all collapse to v_m. In particular, if $k = 0$ and $l = 0$, we must have $x_n = u_0$, and similarly, if $m = 0$ and $n = 0$, we demand that $v_l = u_0$.

The case where $l = 0$ and $n = 0$ was not handled by Fitting in [Fit15]; we give a corresponding rule in Figure 3. In that case, not only do we identify u_k and w_m, but it is also necessary to apply a substitution $\sigma\colon I_\Gamma \to I_\Gamma$ to the indexes in the context $\Gamma^{u_0}\{\ \}$, giving the new context $\sigma\Gamma^{u_0}\{\ \}$, such that $\sigma(u_k) = \sigma(w_m)$ in the whole sequent (and $\sigma(y) = y$ for any other $y \in I_\Gamma$).

For a given set $\mathbb{G} \subseteq \mathbb{N}^4$, we write G to be the set of rules obtained from $\mathbb{G}$ according to Figures 2 and 3. We write $\mathsf{iNK}+\mathsf{G}$ for the system obtained from iNK by adding the rules in G. System $\mathsf{iNK}+\mathsf{G}$ is sound and complete wrt. the corresponding $\mathsf{IK}+\mathsf{G}$ logic. Soundness is proven by Fitting wrt. relational frames; and completeness via a translation to *set-prefixed tableaux system* for which in turn he gives a semantic completeness proof [Fit15].

3 Cut-elimination

In this section, we present a cut-elimination proof for the indexed nested sequent system iNK that relies on a standard double-induction on the height of the derivation above a given cut-rule (Figure 4), and on the depth of the formula A introduced by the cut-rule.

Definition 3.1 The *height of a derivation tree* π, denoted by $\mathsf{ht}(\pi)$, is the length of the longest path in the tree from its root to one of its leaves.

The *depth of a formula* A, denoted by $\mathsf{dp}(A)$, is defined inductively as follows:

$$\begin{array}{lll}\mathsf{dp}(a)=1 & \mathsf{dp}(A\wedge B)=\max(\mathsf{dp}(A),\mathsf{dp}(B))+1 & \mathsf{dp}(\Box A)=\mathsf{dp}(A)+1\\ \mathsf{dp}(\bar{a})=1 & \mathsf{dp}(A\vee B)=\max(\mathsf{dp}(A),\mathsf{dp}(B))+1 & \mathsf{dp}(\Diamond A)=\mathsf{dp}(A)+1\end{array}$$

The *rank* of an instance of cut is the depth of the formula introduced by the cut (read bottom-up). We also write cut_r to denote an instance of cut with rank at most r. The *cut-rank* of a derivation π, denoted by $\mathsf{rk}(\pi)$, is the maximal rank of a cut in π.

To facilitate the overall argument, we consider a variant of system iNK, that we call *system* $\mathsf{i\ddot{N}K}$, that is obtained from iNK by removing the teleportation rule tp (but keeping the bc-rules), and by replacing the id- and $\Diamond$-rules by

$$\ddot{\mathsf{id}}\,\frac{}{\Gamma^{u}\{a\}^{u}\{\bar{a}\}} \quad \text{and} \quad \ddot{\Diamond}\,\frac{\Gamma^{u}\{\Diamond A\}^{u}\{[A,\Delta]\}}{\Gamma^{u}\{\Diamond A\}^{u}\{[\Delta]\}} \tag{5}$$

respectively. The reason behind this is that iNK and $\mathsf{i\ddot{N}K}$ are equivalent (with and without cut, as shown below in Lemma 3.5), but the tp-rule is admissible in the new system, so that we do not need to consider it in the cut-elimination argument.

In the course of the cut-elimination proof we will need some additional structural rules called *weakening*, *contraction*, *necessitation*, and *index substitution* respectively, which are shown in Figure 4. The rules for weakening and contraction are similar to the standard sequent ones except that they can apply deeply inside a context. The rules nec and isub on the other hand cannot be applied deep inside a context; they always work on the whole sequent. In isub, the sequent $\sigma\Gamma$ is obtained from Γ by applying the substitution $\sigma\colon I_\Gamma \to I_\Gamma$ on the indexes occurring in Γ, here σ can be an arbitrary renaming.

Definition 3.2 For a given system $\mathcal{S}$, a rule $\mathsf{r} \notin \mathcal{S}$ with n premisses is *admissible* in $\mathcal{S}$, if for any proofs $\pi_1, \ldots, \pi_n$ of its premisses in $\mathcal{S}$, there is a proof π' of its conclusion in $\mathcal{S}$. Similarly, a rule r is *invertible* in a system $\mathcal{S}$, if for every derivation of the conclusion of r there are derivations for each of its premisses. We say, furthermore, that r is *height* (or *cut-rank*) *preserving* admissible/invertible, if the obtained derivations have at most the same height (resp. at most the same cut-rank) as the original ones.

Lemma 3.3 *Let $\mathbb{G} \subseteq \mathbb{N}^4$ and G the corresponding set of rules.*

1. *The rules nec, w, isub and c are cut-rank and height preserving admissible for $\mathsf{i\ddot{N}K} + \mathsf{G}$.*
2. *All rules of $\mathsf{i\ddot{N}K} + \mathsf{G}$ (except for the axiom $\ddot{\mathsf{id}}$) are cut-rank and height-preserving invertible.*

Proof This proof is analogous to that for the pure nested sequent systems in [Brü09]. The admissibility of nec, $\mathsf{w}^\bullet$ and isub can be shown via induction on the height of the derivation. We can proceed similarly for showing invertibility of the $\wedge^\bullet$, $\vee^\bullet$ and $\Diamond^\bullet$-rules. The inverses of the other rules are just weakenings.

For the admissibility of contraction, we also proceed by induction on the height of the derivation, but we have to make a case analysis on the last rule of this derivation. The cases for $\mathsf{g}_{\mathsf{k,l,m,n}}$ and bc only use the induction hypothesis. For the propositional rules, the $\Box^\circ$- and the $\Diamond^\bullet$-rules, we use invertibility of the rules when the active formula is part of the contracted sequent. And in particular, for the $\ddot{\Diamond}^\circ$-rule (resp. $\ddot{\Box}^\bullet$), we need to distinguish whether the formula $\Diamond^\circ A$ (resp.

$\Box^\bullet$) or the sequent $[{}^w\Delta]$ are part of the contraction, and in some cases, use the height-preserving admissibility of weakening to conclude. $\Box$

Lemma 3.4 *Let $\mathbb{G} \subseteq \mathbb{N}^4$ and G the corresponding set of rules. The rule tp is admissible for $\mathsf{i\ddot{N}K} + \mathsf{G}$ (and for $\mathsf{i\ddot{N}K} + \mathsf{G} + \mathsf{cut}$).*

Proof The proof uses an induction on the number of instances of tp in a proof, eliminating topmost instances first, by an induction on the height of the proof above it and a case analysis of the rule r applied just before tp.

If $\mathsf{r} = \mathsf{\ddot{i}d}$, whether tp applies on the specific atoms or elsewhere in the context, its conclusion has to be itself an axiom, so the stack of $\mathsf{\ddot{i}d}$ and tp can be replaced by a single instance of $\mathsf{\ddot{i}d}$.

$$\mathsf{tp}\,\dfrac{\mathsf{\ddot{i}d}\,\dfrac{}{\Gamma\,{}^u\{a\}\,{}^u\{\bar{a}\}\,{}^u\{\}}}{\Gamma\,{}^u\{a\}\,{}^u\{\}\,{}^u\{\bar{a}\}} \quad\rightsquigarrow\quad \mathsf{\ddot{i}d}\,\dfrac{}{\Gamma\,{}^u\{a\}\,{}^u\{\}\,{}^u\{\bar{a}\}}$$

$$\mathsf{tp}\,\dfrac{\mathsf{\ddot{i}d}\,\dfrac{}{\Gamma\,{}^u\{a\}\,{}^u\{\bar{a}\}}}{\Gamma'\,{}^u\{a\}\,{}^u\{\bar{a}\}} \quad\rightsquigarrow\quad \mathsf{\ddot{i}d}\,\dfrac{}{\Gamma'\,{}^u\{a\}\,{}^u\{\bar{a}\}}$$

If r does not affect the teleported formula, in particular if r is a cut, then the two rules can simply be permuted and we can conclude by applying the induction hypothesis (potentially twice). The same also works if the principal formula is teleported but r is $\ddot{\Diamond}$, or $\mathsf{g_{k,l,m,n}}$, or bc.

$$\mathsf{tp}\,\dfrac{\mathsf{r}\,\dfrac{\Gamma'\,{}^u\{\}\,{}^u\{A\}}{\Gamma\,{}^u\{\}\,{}^u\{A\}}}{\Gamma\,{}^u\{A\}\,{}^u\{\}} \quad\rightsquigarrow\quad \mathsf{r}\,\dfrac{\mathsf{tp}\,\dfrac{\Gamma\,{}^u\{\}\,{}^u\{A\}}{\Gamma'\,{}^u\{A\}\,{}^u\{\}}}{\Gamma\,{}^u\{A\}\,{}^u\{\}}$$

If $\mathsf{r} = \wedge$ or $\vee$, we just need to use the induction hypothesis twice, either in series or in parallel.

$$\mathsf{tp}\,\dfrac{\wedge\,\dfrac{\Gamma\,{}^u\{A\}\,{}^u\{\} \quad \Gamma\,{}^u\{B\}\,{}^u\{\}}{\Gamma\,{}^u\{A \wedge B\}\,{}^u\{\}}}{\Gamma\,{}^u\{\}\,{}^u\{A \wedge B\}} \quad\rightsquigarrow\quad \wedge\,\dfrac{\mathsf{tp}\,\dfrac{\Gamma\,{}^u\{A\}\,{}^u\{\}}{\Gamma\,{}^u\{\}\,{}^u\{A\}} \quad \mathsf{tp}\,\dfrac{\Gamma\,{}^u\{B\}\,{}^u\{\}}{\Gamma\,{}^u\{\}\,{}^u\{B\}}}{\Gamma\,{}^u\{\}\,{}^u\{A \wedge B\}}$$

$$\mathsf{tp}\,\dfrac{\vee\,\dfrac{\Gamma\,{}^u\{A, B\}\,{}^u\{\}}{\Gamma\,{}^u\{A \vee B\}\,{}^u\{\}}}{\Gamma\,{}^u\{\}\,{}^u\{A \vee B\}} \quad\rightsquigarrow\quad \vee\,\dfrac{\mathsf{tp}\,\dfrac{\mathsf{tp}\,\dfrac{\Gamma\,{}^u\{A, B\}\,{}^u\{\}}{\Gamma\,{}^u\{A\}\,{}^u\{B\}}}{\Gamma\,{}^u\{\}\,{}^u\{A, B\}}}{\Gamma\,{}^u\{\}\,{}^u\{A \vee B\}}$$

If $\mathsf{r} = \square$, we transform the derivation as follows and then use the admissibility of weakening (Lemma 3.3) and the induction hypothesis to conclude.

$$\mathsf{tp}\,\dfrac{\square\,\dfrac{\Gamma^{u}\{[^{v}A]\}^{u}\{\}}{\Gamma^{u}\{\square A\}^{u}\{\}}}{\Gamma^{u}\{\}^{u}\{\square A\}} \quad\rightsquigarrow\quad \square\,\dfrac{\mathsf{bc}\,\dfrac{\mathsf{tp}\,\dfrac{\mathsf{w}\,\dfrac{\Gamma^{u}\{[^{v}A]\}^{u}\{\}}{\Gamma^{u}\{[^{v}A]\}^{u}\{[^{v}\ ]\}}}{\Gamma^{u}\{[^{v}\ ]\}^{u}\{[^{v}A]\}}}{\Gamma^{u}\{\}^{u}\{[^{v}A]\}}}{\Gamma^{u}\{\}^{u}\{\square A\}}$$

$\square$

Lemma 3.5 *Let $\mathbb{G} \subseteq \mathbb{N}^4$ and* G *the corresponding set of rules. A sequent Δ is provable in* $\mathsf{iNK} + \mathsf{G}$ *(or in* $\mathsf{iNK} + \mathsf{G} + \mathsf{cut}$*) if and only if it is provable in* $\mathsf{i\ddot{N}K} + \mathsf{G}$ *(resp. in* $\mathsf{i\ddot{N}K} + \mathsf{G} + \mathsf{cut}$*).*

Proof Given a proof of Δ in $\mathsf{iNK} + \mathsf{G}$, we can observe that the rules id and $\Diamond$ are just special cases of the rules $\ddot{\mathsf{id}}$ and $\ddot{\Diamond}$, respectively. Thus, we obtain a proof of Δ in $\mathsf{i\ddot{N}K} + \mathsf{G}$ from admissibility of tp (Lemma 3.4). Conversely, if we have a proof of Δ in $\mathsf{i\ddot{N}K} + \mathsf{G}$, we can obtain a proof of Δ in $\mathsf{iNK} + \mathsf{G}$ by replacing all instance of $\ddot{\mathsf{id}}$ and $\ddot{\Diamond}$ by the following derivations:

$$\mathsf{tp}\,\dfrac{\mathsf{id}\,\dfrac{}{\Gamma^{u}\{\emptyset\}^{u}\{a,\bar{a}\}}}{\Gamma^{u}\{a\}^{u}\{\bar{a}\}} \quad\text{and}\quad \mathsf{tp}\,\dfrac{\Diamond\,\dfrac{\mathsf{tp}\,\dfrac{\Gamma^{u}\{\Diamond A\}^{u}\{[A,\Delta]\}}{\Gamma^{u}\{\emptyset\}^{u}\{\Diamond A,[A,\Delta]\}}}{\Gamma^{u}\{\emptyset\}^{u}\{\Diamond A,[\Delta]\}}}{\Gamma^{u}\{\Diamond A\}^{u}\{[\Delta]\}}$$

respectively. The same proof goes for the system with cut. $\square$

Finally we can prove the main lemma of this section which will correspond to the induction step of the cut-elimination proof.

Lemma 3.6 *If there is a proof π of shape*

$$\mathsf{cut}_{r+1}\,\dfrac{\overset{\pi_1}{\Gamma\{A\}} \qquad \overset{\pi_2}{\Gamma\{\bar{A}\}}}{\Gamma\{\emptyset\}}$$

in $\mathsf{i\ddot{N}K} + \mathsf{G}$ *such that* $\mathsf{rk}(\pi_1) \leq r$ *and* $\mathsf{rk}(\pi_2) \leq r$*, then there is proof π' of $\Gamma\{\emptyset\}$ in* $\mathsf{i\ddot{N}K} + \mathsf{G}$ *such that* $\mathsf{rk}(\pi') \leq r$*.*

Proof We proceed by induction on $\mathsf{ht}(\pi_1) + \mathsf{ht}(\pi_2)$, making a case analysis on the bottommost rules in π_1 and π_2.

1. If π_1 is just an $\ddot{\mathsf{id}}$, there are two sub-cases:

– The cut-formula A is one of the atoms in the identity. Then we have

$$\mathsf{cut}_1\,\dfrac{\mathsf{\ddot{i}d}\,\dfrac{}{\Gamma^{u}\{a\}^{u}\{\bar{a}\}} \qquad \overset{\pi_2}{\Gamma^{u}\{\bar{a}\}^{u}\{\bar{a}\}}}{\Gamma^{u}\{\emptyset\}^{u}\{\bar{a}\}} \quad \rightsquigarrow \quad \mathsf{c}\,\dfrac{\mathsf{tp}\,\dfrac{\overset{\pi_2}{\Gamma^{u}\{\bar{a}\}^{u}\{\bar{a}\}}}{\Gamma^{u}\{\emptyset\}^{u}\{\bar{a},\bar{a}\}}}{\Gamma^{u}\{\emptyset\}^{u}\{\bar{a}\}}$$

where we apply the admissibility of tp (Lemma 3.4) and c (Lemma 3.3).

– If the cut-formula A is not one of the atoms in the identity then we can apply the $\mathsf{\ddot{i}d}$-rule directly to $\Gamma\{\emptyset\}$.

2. If the bottommost rule r of π_1 is bc or $\mathsf{g_{k,l,m,n}}$ then we have

$$\mathsf{cut}_{r+1}\,\dfrac{\mathsf{r}\,\dfrac{\overset{\pi_1'}{\Gamma'\{A\}}}{\Gamma\{A\}} \qquad \overset{\pi_2}{\Gamma\{\bar{A}\}}}{\Gamma\{\emptyset\}} \quad \rightsquigarrow \quad \mathsf{r}\,\dfrac{\mathsf{cut}_{r+1}\,\dfrac{\overset{\pi_1'}{\Gamma'\{A\}} \qquad \mathsf{w}\,\dfrac{\overset{\pi_2}{\Gamma\{\bar{A}\}}}{\Gamma'\{\bar{A}\}}}{\Gamma'\{\emptyset\}}}{\Gamma\{\emptyset\}}$$

and we proceed by induction hypothesis and height-preserving admissibility of weakening (Lemma 3.3).

3. If the bottommost rule r of π_1 is $\mathsf{g_{k,0,m,0}}$ then we have

$$\mathsf{cut}\,\dfrac{\mathsf{g_{k,0,m,0}}\,\dfrac{\overset{\pi_1'}{\sigma_{w\to u}\Gamma\{\Gamma_{k-1}\{[^{u}\Delta]\},\Gamma_{m-1}\{[^{u}\Sigma]\}\}\{A\}}}{\Gamma\{\Gamma_{k-1}\{[^{u}\Delta]\},\Gamma_{m-1}\{[^{w}\Sigma]\}\}\{A\}} \qquad \overset{\pi_2}{\Gamma\{\Gamma_{k-1}\{[^{u}\Delta]\},\Gamma_{m-1}\{[^{w}\Sigma]\}\}\{\bar{A}\}}}{\Gamma\{\Gamma_{k-1}\{[^{u}\Delta]\},\Gamma_{m-1}\{[^{w}\Sigma]\}\}\{\emptyset\}}$$

which can be replaced by

$$\mathsf{g_{k,0,m,0}}\,\dfrac{\mathsf{cut}\,\dfrac{\overset{\pi_1'}{\sigma_{w\to u}\Gamma\{\Gamma_{k-1}\{[^{u}\Delta]\},\Gamma_{m-1}\{[^{w}\Sigma]\}\}\{A\}} \qquad \mathsf{isub}\,\dfrac{\overset{\pi_2}{\Gamma\{\Gamma_{k-1}\{[^{u}\Delta]\},\Gamma_{m-1}\{[^{w}\Sigma]\}\}\{\bar{A}\}}}{\sigma_{w\to u}\Gamma\{\Gamma_{k-1}\{[^{u}\Delta]\},\Gamma_{m-1}\{[^{u}\Sigma]\}\}\{\bar{A}\}}}{\sigma_{w\to u}\Gamma\{\Gamma_{k-1}\{[^{u}\Delta]\},\Gamma_{m-1}\{[^{u}\Sigma]\}\}\{\emptyset\}}}{\Gamma\{\Gamma_{k-1}\{[^{u}\Delta]\},\Gamma_{m-1}\{[^{w}\Sigma]\}\}\{\emptyset\}}$$

where $\Gamma_{k-1}\{\ \}$ and $\Gamma_{m-1}\{\ \}$ correspond to contexts of the form $[^{u_1}\Delta_1,\dots[^{u_{k-1}}\Delta_{k-1},\{\ \}]]$ and $[^{w_1}\Sigma_1,\dots[^{w_m}\Sigma_m,\{\ \}]]$ respectively, and we can proceed by induction hypothesis.

4. If the bottommost rule r of π_1 is one of $\wedge$, $\vee$, $\Box$, or $\ddot{\Diamond}$, such that the principal formula of r is not the cut-formula A, then we proceed as in the previous case; we apply the height-preserving invertibility (Lemma 3.3) of the rules $\wedge$, $\vee$, $\Box$, or $\ddot{\Diamond}$ (and apply it twice in the case of the $\wedge$-rule as illustrated below) and proceed by induction hypothesis. (We write r^{-1} to denote the admissible inverse of the rule r.)

$$\mathsf{cut}_{r+1}\,\dfrac{\wedge\,\dfrac{\begin{array}{c}\pi_1'\\ \Gamma\{A\}\{C\}\end{array} \quad \begin{array}{c}\pi_1''\\ \Gamma\{B\}\{C\}\end{array}}{\Gamma\{A\wedge B\}\{C\}} \qquad \begin{array}{c}\pi_2\\ \Gamma\{A\wedge B\}\{\bar{C}\}\end{array}}{\Gamma\{A\wedge B\}\{\emptyset\}} \qquad \text{is reduced to}$$

$$\wedge\,\dfrac{\mathsf{cut}_{r+1}\,\dfrac{\begin{array}{c}\pi_1'\\ \Gamma\{A\}\{C\}\end{array} \quad \wedge^{-1}\,\dfrac{\begin{array}{c}\pi_2\\ \Gamma\{A\wedge B\}\{\bar{C}\}\end{array}}{\Gamma\{A\}\{\bar{C}\}}}{\Gamma\{A\}\{\emptyset\}} \qquad \mathsf{cut}_{r+1}\,\dfrac{\begin{array}{c}\pi_1''\\ \Gamma\{B\}\{C\}\end{array} \quad \wedge^{-1}\,\dfrac{\begin{array}{c}\pi_2\\ \Gamma\{A\wedge B\}\{\bar{C}\}\end{array}}{\Gamma\{B\}\{\bar{C}\}}}{\Gamma\{B\}\{\emptyset\}}}{\Gamma\{A\wedge B\}\{\emptyset\}}$$

The cases 1-3 are similar for the bottommost rule of π_2. Let us now consider the non-axiomatic key cases:

5. The bottommost rules r_1 of π_1 and r_2 of π_2 are among $\wedge$, $\vee$, $\Box$, or $\ddot{\Diamond}$, and for both the cut-formula is principal. Then we have the following cases:

   – $A = B \vee C$: Then we have
   $$\mathsf{cut}_{r+1}\,\dfrac{\vee\,\dfrac{\begin{array}{c}\pi_1'\\ \Gamma\{B,C\}\end{array}}{\Gamma\{B\vee C\}} \qquad \wedge\,\dfrac{\begin{array}{c}\pi_2'\\ \Gamma\{\bar{B}\}\end{array} \quad \begin{array}{c}\pi_2''\\ \Gamma\{\bar{C}\}\end{array}}{\Gamma\{\bar{B}\wedge\bar{C}\}}}{\Gamma\{\emptyset\}}$$
   which is reduced to
   $$\mathsf{cut}_r\,\dfrac{\mathsf{cut}_r\,\dfrac{\begin{array}{c}\pi_1'\\ \Gamma\{B,C\}\end{array} \quad \mathsf{w}\,\dfrac{\begin{array}{c}\pi_2'\\ \Gamma\{\bar{B}\}\end{array}}{\Gamma\{\bar{B},C\}}}{\Gamma\{C\}} \qquad \begin{array}{c}\pi_2''\\ \Gamma\{\bar{C}\}\end{array}}{\Gamma\{\emptyset\}}$$
   where we can apply the height-preserving admissibility of weakening.

   – $A = B \wedge C$: Similar.

– $A = \Diamond B$. Then we have

$$\mathsf{cut}_{r+1}\,\dfrac{\ddot{\Diamond}\,\dfrac{\dfrac{\pi_1'}{\Gamma^{w}\{\Diamond B\}^{w}\{[^{u}B, \Delta]\}}}{\Gamma^{w}\{\Diamond B\}^{w}\{[^{u}\Delta]\}} \qquad \Box\,\dfrac{\dfrac{\pi_2'}{\Gamma^{w}\{[^{v}\bar{B}]\}^{w}\{[^{u}\Delta]\}}}{\Gamma^{w}\{\Box\bar{B}\}^{w}\{[^{u}\Delta]\}}}{\Gamma^{w}\{\emptyset\}^{w}\{[^{u}\Delta]\}} \quad \rightsquigarrow$$

which can be reduced to

$$\mathsf{cut}_{r}\,\dfrac{\mathsf{cut}_{r+1}\,\dfrac{\dfrac{\pi_1'}{\Gamma^{w}\{\Diamond B\}^{w}\{[^{u}B, \Delta]\}} \qquad \Box\,\dfrac{\mathsf{w}\,\dfrac{\dfrac{\pi_2'}{\Gamma^{w}\{[^{v}\bar{B}]\}^{w}\{[^{u}\Delta]\}}}{\Gamma^{w}\{[^{v}\bar{B}]\}^{w}\{[^{u}B, \Delta]\}}}{\Gamma^{w}\{\Box\bar{B}\}^{w}\{[^{u}B, \Delta]\}}}{\Gamma^{w}\{\emptyset\}^{w}\{[^{u}B, \Delta]\}} \qquad \mathsf{bc}\,\dfrac{\mathsf{tp}\,\dfrac{\mathsf{isub}\,\dfrac{\dfrac{\pi_2'}{\Gamma^{w}\{[^{v}\bar{B}]\}^{w}\{[^{u}\Delta]\}}}{\Gamma^{w}\{[^{u}\bar{B}]\}^{w}\{[^{u}\Delta]\}}}{\Gamma^{w}\{[^{u}\ ]\}^{w}\{[^{u}\bar{B}, \Delta]\}}}{\Gamma^{w}\{\emptyset\}^{w}\{[^{u}\bar{B}, \Delta]\}}}{\Gamma^{w}\{\emptyset\}^{w}\{[^{u}\Delta]\}}$$

where on the left branch we use height-preserving admissibility of weakening and proceed by induction hypothesis, and on the right branch we use admissibility of the isub- and tp-rules (Lemmas 3.3 and 3.4).

– $A = \Box B$. Similar.

□

Theorem 3.7 *If a sequent Γ is derivable in* $\mathsf{i\ddot{N}K} + \mathsf{G} + \mathsf{cut}$ *then it is also derivable in* $\mathsf{i\ddot{N}K} + \mathsf{G}$.

Proof We show that if there exists a proof π of Γ in $\mathsf{i\ddot{N}K} + \mathsf{G} + \mathsf{cut}$ then Γ is also derivable in $\mathsf{i\ddot{N}K} + \mathsf{G}$, by an induction on the cut rank of π. The induction step uses also an induction on the number of occurrences of cut with the maximal rank as well as Lemma 3.6 to eliminate each time the topmost occurrence in the proof. □

Corollary 3.8 *If a sequent Γ is derivable in* $\mathsf{iNK} + \mathsf{G} + \mathsf{cut}$ *then it is also derivable in* $\mathsf{iNK} + \mathsf{G}$.

Proof Following Theorem 3.7 and Lemma 3.5. □

4 From classical to intuitionistic

Starting from the proof system for classical modal logic discussed in the previous section, we will show now how to obtain an intuitionistic variant. This will be done in a similar way as Gentzen did in his original work for the ordinary sequent calculus [Gen34].

The first step is to enrich the language of formulas with a negation, such that they are no longer restricted to negative normal form. Since we want to get an intuitionistic system as well, we also include implication, bottom, and top as primitives here:

$$A, B \ldots ::= a \mid \bot \mid \top \mid \neg A \mid A \wedge B \mid A \vee B \mid A \supset B \mid \Box A \mid \Diamond A \tag{6}$$

Intuitionistic modal logic IK is obtained from intuitionistic propositional logic by adding the axioms

$$\begin{array}{ll} \mathsf{k}_1: \ \Box(A \supset B) \supset (\Box A \supset \Box B) & \mathsf{k}_3: \ \Diamond(A \vee B) \supset (\Diamond A \vee \Diamond B) \\ \mathsf{k}_2: \ \Box(A \supset B) \supset (\Diamond A \supset \Diamond B) & \mathsf{k}_4: \ (\Diamond A \supset \Box B) \supset \Box(A \supset B) \\ & \mathsf{k}_5: \ \Diamond \bot \supset \bot \end{array} \tag{7}$$

and the rule nec, similarly to Section 2. These axioms are logical consequences of k in the classical case but not in the intuitionistic case.[2]

We will consider the following schema as the intuitionistic equivalent to Scott-Lemmon axioms:

$$\mathsf{g}_{\mathsf{k,l,m,n}}: (\Diamond^k \Box^l A \supset \Box^m \Diamond^n A) \wedge (\Diamond^m \Box^n A \supset \Box^k \Diamond^l A) \tag{8}$$

The two conjuncts correspond to the classical $\mathsf{g}_{\mathsf{k,l,m,n}}$ and $\mathsf{g}_{\mathsf{m,n,k,l}}$ which are equivalent via De Morgan dualities in classical logic, but not in intuitionistic modal logic.

In the following, we will first present a *two-sided* version of the classical one-sided system iNK that was given in Figure 1. For this, the first step is to include the distinction between *input* and *output* formulas into the data structure. (similarly to the distinction between"left of the turnstile" and "right of the turnstile" in Gentzen sequent calculi). However, since we have no "turnstile" in nested sequents, we cannot simply write formulas on the left or on the right of it. To that purpose we use here the notion of *polarity*, as studied by Lamarche in [Lam01]. We assign to every formula in the nested sequent a unique polarity: either *input*, denoted by a $\bullet$-superscript and analogous to "left of the turnstile if there was a turnstile", or *output*, denoted by a $\circ$-superscript and analogous to "right of the turnstile if there was a turnstile". A two-sided indexed nested sequent therefore is of the following form, denoted by Γ° if it contains at least one input formula and by $\Lambda^\bullet$ otherwise:

$$\begin{array}{lll} \Gamma^\circ & ::= & \Lambda^\bullet \mid \Gamma^\circ, A^\circ \mid \Gamma^\circ, [^w \Gamma^\circ] \\ \Lambda^\bullet & ::= & \emptyset \ \ \mid \Lambda^\bullet, B^\bullet \mid \Lambda^\bullet, [^u \Lambda^\bullet] \end{array} \tag{9}$$

We are now ready to see the inference rules. The two-sided version of iNK_2 is shown in Figure 5. As expected, the rules for output formulas are the same as

[2] This is the variant of IK first mentioned in [FS84] and [PS86] and studied in detail in [Sim94]. There are many more variants of intuitionistic modal logic, e.g. [Fit48,Pra65,BdP00,PD01]. Another popular variant is *constructive modal logic* (e.g. [MS11]), which rejects axioms k_3-k_5 in (7) and only allows k_1 and k_2. It has a different cut-elimination proof in nested sequents [ADS15]. For this reason we concentrate in this paper on IK which allows all of k_1-k_5.

$$\bot^\bullet\ \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{id}\ \frac{}{\Gamma\{a^\bullet, a^\circ\}} \qquad \top^\circ\ \frac{}{\Gamma\{\top^\circ\}}$$

$$\wedge^\bullet\ \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ\ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}} \qquad \vee^\bullet\ \frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee^\circ_c\ \frac{\Gamma\{A^\circ, B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\neg^\bullet_c\ \frac{\Gamma\{A^\circ\}}{\Gamma\{\neg A^\bullet\}} \qquad \neg^\circ_c\ \frac{\Gamma\{A^\bullet\}}{\Gamma\{\neg A^\circ\}} \qquad \supset^\bullet_c\ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \qquad \supset^\circ\ \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box^\bullet\ \frac{\Gamma\{\Box A^\bullet, [^w A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [^w \Delta]\}} \qquad \Box^\circ\ \frac{\Gamma\{[^v A^\circ]\}}{\Gamma\{\Box A^\circ\}} \qquad \Diamond^\bullet\ \frac{\Gamma\{[^v A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Diamond^\circ_c\ \frac{\Gamma\{\Diamond A^\circ, [^w A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [^w \Delta]\}}$$

$$\mathsf{tp}\ \frac{\Gamma^{\,w}\{\emptyset\}^{\,w}\{A\}}{\Gamma^{\,w}\{A\}^{\,w}\{\emptyset\}} \qquad \mathsf{bc}_1\ \frac{\Gamma^{\,w}\{[^u \Delta]\}^{\,w}\{[^u \emptyset]\}}{\Gamma^{\,w}\{[^u \Delta]\}^{\,w}\{\emptyset\}} \qquad \mathsf{bc}_2\ \frac{\Gamma^{\,w}\{[^u \Delta^{\,w}\{[^u \emptyset]\}]\}}{\Gamma^{\,w}\{[^u \Delta^{\,w}\{\emptyset\}]\}}$$

Fig. 5. System iNK_2

in the one-sided case, and the rules for input formulas show dual behavior. The negation rules flip the polarity, which is similar to ordinary sequent calculus, where the negation rules move a formula to the other side of the turnstile.

Finally, the step from classical to intuitionistic simply consists in restricting the number of output formulas in the sequent to one. Again, this is as in ordinary Gentzen sequent calculus [Gen34], but it is crucial to observe that we count the whole sequent, and not every bracket separately [Str13]. So an intuitionistic indexed nested sequent is of the form:

$$\Gamma^\circ \ ::= \ \Lambda^\bullet, A^\circ \mid \Lambda^\bullet, [^v \Gamma^\circ] \tag{10}$$

where $\Lambda^\bullet$ is defined as in (9).

Since we do not have an explicit contraction rule, but have contraction incorporated into inference rules (e.g., $\Box^\bullet$), some of the inference rules have to be slightly changed, namely $\vee^\circ$, $\supset^\bullet$ and $\Diamond^\circ$ in order to maintain the property that each sequent in a proof contains exactly one output formula. In particular, to ensure that both premisses of the $\supset^\bullet$-rule are intuitionistic sequents, the notation $\Gamma^\downarrow\{\ \}$ stands for the context obtained from $\Gamma\{\ \}$ by removing the output formula. The intuitionistic system is shown in Figure 6. Observe that the structural rules in the bottom line of each figure are identical for all three systems (one-sided classical, two-sided classical, and two-sided intuitionistic).

Finally, the cut-elimination proof conducted in $\mathsf{iNK} + \mathsf{G}$ can be reproduced in a similar fashion in the classical and the intuitionistic two-sided systems, the two-sided cut-rule being of the form $\mathsf{cut}_c\ \dfrac{\Gamma\{A^\circ\} \quad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}}$ in the classical case, and $\mathsf{cut}_i\ \dfrac{\Gamma^\downarrow\{A^\circ\} \quad \Gamma\{A^\bullet\}}{\Gamma\{\emptyset\}}$ in the intuitionistic case, as there a unique output formula needs to be maintained in the left branch.

$$\bot^\bullet\ \frac{}{\Gamma\{\bot^\bullet\}} \qquad \mathsf{id}\ \frac{}{\Gamma\{a^\bullet, a^\circ\}} \qquad \top^\circ\ \frac{}{\Gamma\{\top^\circ\}}$$

$$\wedge^\bullet\ \frac{\Gamma\{A^\bullet, B^\bullet\}}{\Gamma\{A \wedge B^\bullet\}} \qquad \wedge^\circ\ \frac{\Gamma\{A^\circ\} \quad \Gamma\{B^\circ\}}{\Gamma\{A \wedge B^\circ\}} \qquad \vee^\bullet\ \frac{\Gamma\{A^\bullet\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \vee B^\bullet\}} \qquad \vee_1^\circ\ \frac{\Gamma\{A^\circ\}}{\Gamma\{A \vee B^\circ\}} \qquad \vee_2^\circ\ \frac{\Gamma\{B^\circ\}}{\Gamma\{A \vee B^\circ\}}$$

$$\neg^\bullet\ \frac{\Gamma^\downarrow\{\neg A^\bullet, A^\circ\}}{\Gamma\{\neg A^\bullet\}} \qquad \neg^\circ\ \frac{\Gamma\{A^\bullet, \bot^\circ\}}{\Gamma\{\neg A^\circ\}} \qquad \supset^\bullet\ \frac{\Gamma^\downarrow\{A \supset B^\bullet, A^\circ\} \quad \Gamma\{B^\bullet\}}{\Gamma\{A \supset B^\bullet\}} \qquad \supset^\circ\ \frac{\Gamma\{A^\bullet, B^\circ\}}{\Gamma\{A \supset B^\circ\}}$$

$$\Box^\bullet\ \frac{\Gamma\{\Box A^\bullet, [^w A^\bullet, \Delta]\}}{\Gamma\{\Box A^\bullet, [^w \Delta]\}} \qquad \Box^\circ\ \frac{\Gamma\{[^v A^\circ]\}}{\Gamma\{\Box A^\circ\}} \qquad \Diamond^\bullet\ \frac{\Gamma\{[^v A^\bullet]\}}{\Gamma\{\Diamond A^\bullet\}} \qquad \Diamond_i^\circ\ \frac{\Gamma\{[^w A^\circ, \Delta]\}}{\Gamma\{\Diamond A^\circ, [^w \Delta]\}}$$

$$\mathsf{tp}\ \frac{\Gamma^{\,w}\{\emptyset\}^{\,w}\{A\}}{\Gamma^{\,w}\{A\}^{\,w}\{\emptyset\}} \qquad \mathsf{bc}_1\ \frac{\Gamma^{\,w}\{[^u \Delta]\}^{\,w}\{[^u \emptyset]\}}{\Gamma^{\,w}\{[^u \Delta]\}^{\,w}\{\emptyset\}} \qquad \mathsf{bc}_2\ \frac{\Gamma^{\,w}\{[^u \Delta^{\,w}\{[^u \emptyset]\}]\}}{\Gamma^{\,w}\{[^u \Delta^{\,w}\{\emptyset\}]\}}$$

Fig. 6. System iNIK

Theorem 4.1 *Let* $\mathbb{G} \subseteq \mathbb{N}^4$ *and* G *be the corresponding set of rules. If a sequent* Γ *is derivable in* $\mathsf{iNK}_2 + \mathsf{G} + \mathsf{cut}_c$ *(resp.* $\mathsf{iNIK} + \mathsf{G} + \mathsf{cut}_i$*) then it is also derivable in* $\mathsf{iNK}_2 + \mathsf{G}$ *(resp.* $\mathsf{iNIK} + \mathsf{G}$*).*

Proof The proof works similarly to the one of Theorem 3. We need to transform the two-sided systems in a similar fashion as we did with $\mathsf{i\ddot{N}K}$, removing the tp-rule and changing the rules id, $\Box^\bullet$, $\Diamond_c^\circ$ and $\Diamond^\circ$ for respectively

$$\mathsf{\ddot{i}d}\ \frac{}{\Gamma^{\,u}\{a^\circ\}^{\,u}\{a^\bullet\}} \qquad \ddot{\Box}^\bullet\ \frac{\Gamma^{\,u}\{\Box A^\bullet\}^{\,u}\{[A^\bullet, \Delta]\}}{\Gamma^{\,u}\{\Box A^\bullet\}^{\,u}\{[\Delta]\}}$$

$$\ddot{\Diamond}_c^\circ\ \frac{\Gamma^{\,u}\{\Diamond A^\circ\}^{\,u}\{[A^\circ, \Delta]\}}{\Gamma^{\,u}\{\Diamond A^\circ\}^{\,u}\{[\Delta]\}} \qquad \ddot{\Diamond}^\circ\ \frac{\Gamma^{\,u}\{\emptyset\}^{\,u}\{[A^\circ, \Delta]\}}{\Gamma^{\,u}\{\Diamond A^\circ\}^{\,u}\{[\Delta]\}}$$

Then, we can easily extend Lemma 3.3 and 3.4 to the two-sided setting. And so, we can prove a reduction lemma like Lemma 3.6 for the two-sided classical and intuitionistic systems. That is, for a derivation π of the form

$$\mathsf{cut}_{r+1}\ \frac{\overset{\pi_1}{\Gamma\{A^\circ\}} \quad \overset{\pi_2}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}} \qquad \text{or} \qquad \mathsf{cut}_{r+1}\ \frac{\overset{\pi_1}{\Gamma^\downarrow\{A^\circ\}} \quad \overset{\pi_2}{\Gamma\{A^\bullet\}}}{\Gamma\{\emptyset\}}$$

such that $\mathsf{rk}(\pi_1) \leq r$ and $\mathsf{rk}(\pi_2) \leq r$, there is derivation π' of $\Gamma\{\emptyset\}$ such that $\mathsf{rk}(\pi') \leq r$. The proof is almost identical, except that the reduction cases now occur between the left and the right rule for each connective. We proceed by induction on $\mathsf{ht}(\pi_1) + \mathsf{ht}(\pi_2)$, making a case analysis on the bottommost rules in π_1 and π_2. If the cut-formula is not active in the bottommost rules in π_1 and

π_2 (this includes the case of the $\mathsf{g}_{\mathsf{k,l,m,n}}$ rules), then we use height-preserving invertibility and proceed by the induction hypothesis. For the cases in which the cut-formulas are active in both branches, we only show two examples, as other cases are simpler or just similar to those for the pure nested sequent systems [Brü09,Str13].

– If we have an $\ddot{\mathsf{i}}\mathsf{d}$ on one side, there are two cases:

$$\mathsf{cut}_1\,\dfrac{\mathsf{id}\,\dfrac{}{\Gamma^{\downarrow}\,{}^{v}\{a^\bullet\}\,{}^{v}\{a^\circ\}} \qquad \overset{\pi_2}{\Gamma\,{}^{v}\{a^\bullet\}\,{}^{v}\{a^\bullet\}}}{\Gamma\,{}^{v}\{a^\bullet\}\,{}^{v}\{\emptyset\}} \quad\rightsquigarrow\quad \mathsf{c}\,\dfrac{\mathsf{tp}\,\dfrac{\overset{\pi_2}{\Gamma\,{}^{v}\{a^\bullet\}\,{}^{v}\{a^\bullet\}}}{\Gamma\,{}^{v}\{a^\bullet, a^\bullet\}\,{}^{v}\{\emptyset\}}}{\Gamma\,{}^{v}\{a^\bullet\}\,{}^{v}\{\emptyset\}}$$

and

$$\mathsf{cut}_1\,\dfrac{\overset{\pi_1}{\Gamma\,{}^{v}\{a^\circ\}\,{}^{v}\{\emptyset\}} \qquad \mathsf{id}\,\dfrac{}{\Gamma\,{}^{v}\{a^\bullet\}\,{}^{v}\{a^\circ\}}}{\Gamma\,{}^{v}\{\emptyset\}\,{}^{v}\{a^\circ\}} \quad\rightsquigarrow\quad \mathsf{tp}\,\dfrac{\overset{\pi_1}{\Gamma\,{}^{v}\{a^\circ\}\,{}^{v}\{\emptyset\}}}{\Gamma\,{}^{v}\{\emptyset\}\,{}^{v}\{a^\circ\}}$$

where we use the cut-rank preserving admissibility of tp and c.

– The $\Box^\circ$-$\ddot{\Box}^\bullet$ case is reduced as follows

$$\mathsf{cut}_{r+1}\,\dfrac{\Box^\circ\,\dfrac{\overset{\pi_1}{\Gamma\,{}^{w}\{[{}^{v}B^\circ]\}\,{}^{w}\{[{}^{u}\Delta]\}}}{\Gamma\,{}^{w}\{\Box B^\circ\}\,{}^{w}\{[{}^{u}\Delta]\}} \qquad \ddot{\Box}^\bullet\,\dfrac{\overset{\pi_2}{\Gamma\,{}^{w}\{\Box^\bullet B\}\,{}^{w}\{[{}^{u}B^\bullet,\Delta]\}}}{\Gamma\,{}^{w}\{\Box^\bullet B\}\,{}^{w}\{[{}^{u}\Delta]\}}}{\Gamma\,{}^{w}\{\emptyset\}\,{}^{w}\{[{}^{u}\Delta]\}} \quad\rightsquigarrow$$

$$\mathsf{cut}_{r}\,\dfrac{\mathsf{bc}\,\dfrac{\mathsf{tp}\,\dfrac{\mathsf{isub}\,\dfrac{\overset{\pi_1}{\Gamma^{\downarrow}\,{}^{w}\{[{}^{v}B^\circ]\}\,{}^{w}\{[{}^{u}\Delta]\}}}{\Gamma^{\downarrow}\,{}^{w}\{[{}^{u}B^\circ]\}\,{}^{w}\{[{}^{u}\Delta]\}}}{\Gamma^{\downarrow}\,{}^{w}\{[{}^{u}\ ]\}\,{}^{w}\{[{}^{u}B^\circ,\Delta]\}}}{\Gamma^{\downarrow}\,{}^{w}\{\emptyset\}\,{}^{w}\{[{}^{u}B^\circ,\Delta]\}} \qquad \mathsf{cut}_{r+1}\,\dfrac{\Box\,\dfrac{\mathsf{w}\,\dfrac{\overset{\pi_1}{\Gamma^{\downarrow}\,{}^{w}\{[{}^{v}B^\circ]\}\,{}^{w}\{[{}^{u}\Delta]\}}}{\Gamma^{\downarrow}\,{}^{w}\{[{}^{v}B^\circ]\}\,{}^{w}\{[{}^{u}B^\bullet,\Delta]\}}}{\Gamma^{\downarrow}\,{}^{w}\{\Box B^\circ\}\,{}^{w}\{[{}^{u}B^\bullet,\Delta]\}} \quad \overset{\pi_2}{\Gamma\,{}^{w}\{\Box B^\bullet\}\,{}^{w}\{[{}^{u}B^\bullet,\Delta]\}}}{\Gamma\,{}^{w}\{\emptyset\}\,{}^{w}\{[{}^{u}B^\bullet,\Delta]\}}}{\Gamma\,{}^{w}\{\emptyset\}\,{}^{w}\{[{}^{u}\Delta]\}}$$

and we use the cut-rank and height preserving admissibility of w and the cut-rank preserving admissibility of tp and isub.

Finally, we can prove a similar result as Lemma 3.5 in the two-sided setting to complete the proof. □

The cut-elimination theorem can be used to show that every theorem of $\mathsf{K}+\mathsf{G}$ (resp. $\mathsf{IK}+\mathsf{G}$) is a theorem of $\mathsf{iNK}_2+\mathsf{G}$ (resp. $\mathsf{iNIK}+\mathsf{G}$).

Theorem 4.2 *Let $\mathbb{G}\subseteq\mathbb{N}^4$ and G denote at the same time the correponding set of axioms and of nested sequent rules. If A is provable in the Hilbert system $\mathsf{IK}+\mathsf{G}$, then the sequent A° is provable in the indexed nested sequent system $\mathsf{iNIK}+\mathsf{G}$.*

Proof The axioms of intuitionistic propositional logic as well as the axioms k_1-k_5 can be derived in iNIK, in the same way as in the usual nested sequent system presented in [Str13]. The inference rule nec can be simulated by the structural rule nec, which is admissible in $\mathsf{iNIK} + \mathsf{G}$ (Lemma 3.3), and *modus ponens* mp can be simulated by the cut-rule, which is also admissible (Theorem 4.1). Thus, it remains to show that any $\mathsf{g}_{\mathsf{k,l,m,n}}$ axiom can derived, using the corresponding $\mathsf{g}_{\mathsf{k,l,m,n}}$-rule:

$$
\supset^\circ \dfrac{k{\cdot}\Diamond^\bullet, m{\cdot}\Box^\circ \dfrac{\mathsf{g}_{\mathsf{k,l,m,n}} \dfrac{l{\cdot}\Box^\bullet, n{\cdot}\Diamond^\circ \dfrac{\mathsf{tp} \dfrac{\mathsf{id} \dfrac{}{\begin{array}{r}[^{u_1}\dots[^{u_k}\Box^l p^\bullet,[^{v_1}\Box^{l-1}p^\bullet,\dots[^{v_{l-1}}\Box p^\bullet,[^{v_l}]]\dots]]\dots],\\ [^{w_1}\dots[^{w_m}[^{x_1}\dots[^{x_{n-1}}[^{x_n}p^\bullet,p^\circ]]\dots]]\dots]\end{array}}}{\begin{array}{r}[^{u_1}\dots[^{u_k}\Box^l p^\bullet,[^{v_1}\Box^{l-1}p^\bullet,\dots[^{v_{l-1}}\Box p^\bullet,[^{v_l}p^\bullet]]\dots]]\dots],\\ [^{w_1}\dots[^{w_m}[^{x_1}\dots[^{x_{n-1}}[^{x_n}p^\circ]]\dots]]\dots]\end{array}}\; c_l = d_n}{[^{u_1}\dots[^{u_k}\ \Box^l p^\bullet,[^{v_1}\dots[^{v_l}]\dots]]\dots],[^{w_1}\dots[^{w_m}\ \Diamond^n p^\circ,[^{x_1}\dots[^{x_n}]\dots]]\dots]}}{[^{u_1}\dots[^{u_k}\ \Box^l p^\bullet]\dots],[^{w_1}\dots[^{w_m}\ \Diamond^n p^\circ]\dots]}}{\Diamond^k\Box^l p^\bullet, \Box^m\Diamond^n p^\circ}}{\Diamond^k\Box^l p \supset \Box^m\Diamond^n p^\circ}
$$

□

Note that, in the classical case, a similar proof can be conducted, and provides an alternative to the completeness of indexed nested sequents wrt. set prefixed tableaux in [Fit15].

Theorem 4.3 *Let $\mathbb{G} \subseteq \mathbb{N}^4$ and G denote at the same time the correponding set of axioms and of nested sequent rules. Let A be a modal formula. If A is provable in the Hilbert system $\mathsf{K}+\mathsf{G}$, then the sequent A° is provable in the indexed nested sequent system $\mathsf{iNK}_2 + \mathsf{G}$.*

However, there are examples of theorems of $\mathsf{iNIK}+\mathsf{G}$ that are not theorems of $\mathsf{IK}+\mathsf{G}$, that is, the indexed nested sequent system is not sound with respect to the Hilbert axiomatisation using what we gave above as the intuitionistic alternative to Scott-Lemmon axioms. There is already a simple counter-example when one considers G to be composed with only the axiom $\mathsf{g}_{1,1,1,1} : \Diamond\Box A \supset \Box\Diamond A$. Then, the formula

$$
F = (\Diamond(\Box(a \vee b) \wedge \Diamond a) \wedge \Diamond(\Box(a \vee b) \wedge \Diamond b)) \supset \Diamond(\Diamond a \wedge \Diamond b) \qquad (11)
$$

is derivable in $\mathsf{iNIK} + \mathsf{g}_{1,1,1,1}$ (see Figure 7), but is not a theorem of $\mathsf{IK} + \mathsf{g}_{1,1,1,1}$ (as mentioned in [Sim94]). Thus, the logic given by the Hilbert axiomatisation $\mathsf{IK}+\mathsf{G}$ and the one given by the indexed nested sequent system $\mathsf{iNIK}+\mathsf{G}$ actually differ in the intuitionistic case. We will address this issue in more details in the next section.

5 Semantics of the Scott-Lemmon axioms

In the classical case, the indexed nested sequent system is not only equivalent to the Hilbert axiomatisation using Scott-Lemmon axioms, it is actually sound and

$$
\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c}\pi_1\\ \vdots\\ {[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z a^\bullet]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z]],\Diamond(\Diamond a\wedge\Diamond b)^\circ}\end{array}\qquad \begin{array}{c}\pi_2\\ \vdots\\ {[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z b^\bullet]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z]],\Diamond(\Diamond a\wedge\Diamond b)^\circ}\end{array}}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z]],\Diamond(\Diamond a\wedge\Diamond b)^\circ}\;\Box^\bullet,\vee^\bullet}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet]],\Diamond(\Diamond a\wedge\Diamond b)^\circ}\;\mathsf{g}_{1,1,1,1}}{\Diamond(\Box(a\vee b)\wedge\Diamond a)^\bullet,\Diamond(\Box(a\vee b)\wedge\Diamond b)^\bullet,\Diamond(\Diamond a\wedge\Diamond b)^\circ}\;\Diamond^\bullet,\wedge^\bullet,\Diamond^\bullet}{(\Diamond(\Box(a\vee b)\wedge\Diamond a)\wedge\Diamond(\Box(a\vee b)\wedge\Diamond b))\supset\Diamond(\Diamond a\wedge\Diamond b)^\circ}\;\supset^\circ,\wedge^\bullet
$$

$$
\pi_1 \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z a^\bullet,a^\circ]]}\;\mathsf{id}}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z a^\bullet],\Diamond a^\circ]}\;\Diamond^\circ \qquad \dfrac{\dfrac{}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet,b^\circ],[^z a^\bullet]]}\;\mathsf{id}}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z a^\bullet],\Diamond b^\circ]}\;\Diamond^\circ}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z a^\bullet],\Diamond a\wedge\Diamond b^\circ]}\;\wedge^\circ}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z a^\bullet]],\Diamond(\Diamond a\wedge\Diamond b)^\circ}\;\Diamond^\circ}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z a^\bullet]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z]],\Diamond(\Diamond a\wedge\Diamond b)^\circ}\;\mathsf{tp}
$$

$$
\pi_2 \quad = \quad \dfrac{\dfrac{\dfrac{\dfrac{}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet,a^\circ],[^z b^\bullet]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z]]}\;\mathsf{id}}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z b^\bullet],\Diamond a^\circ],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z]]}\;\Diamond^\circ \qquad \dfrac{\dfrac{}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z b^\bullet,b^\circ]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z]]}\;\mathsf{id}}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z b^\bullet],\Diamond b^\circ],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z]]}\;\Diamond^\circ}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z b^\bullet],\Diamond a\wedge\Diamond b^\circ],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z]]}\;\wedge^\circ}{[^u\Box(a\vee b)^\bullet,[^x a^\bullet],[^z b^\bullet]],[^v\Box(a\vee b)^\bullet,[^y b^\bullet],[^z]],\Diamond(\Diamond a\wedge\Diamond b)^\circ}\;\Diamond^\circ
$$

Fig. 7. Derivation of F in $\mathsf{iNIK}+\mathsf{g}_{1,1,1,1}$

complete wrt. the corresponding Kripke semantics. In this section, we investigate the behavior of system iNIK, and its extension, regarding the Kripke semantics for intuitionistic modal logics.

Let us therefore briefly recall the standard Kripke semantics of classical and intuitionistic modal logics. A *classical frame* $\langle W, R\rangle$ is defined as a non-empty set W of *worlds* and a binary relation $R \subseteq W \times W$, called the *accessibility relation*. An *intuitionistic frame* $\langle W, R, \leq\rangle$ is additionally equipped with a preorder $\leq$ on W, such that:

(F1) For all $u, v, v' \in W$, if uRv and $v \leq v'$, there exists $u' \in W$ such that $u \leq u'$ and $u'Rv'$.

(F2) For all $u', u, v \in W$, if $u \leq u'$ and uRv, there exists $v' \in W$ such that $u'Rv'$ and $v \leq v'$.

A *classical model* $\mathcal{M} = \langle W, R, V\rangle$ is a classical frame together with a *valuation* function $V\colon W \to 2^{\mathcal{A}}$ mapping each world w to the set of propositional variables which are true in w. In an *intuitionistic model* $\langle W, R, \leq, V\rangle$, the function V must be monotone with respect to $\leq$, i.e. $w \leq v$ implies $V(w) \subseteq V(v)$.

We write $w \Vdash a$ if $a \in V(w)$. From there, the relation $\Vdash$ is extended to all formulas in a parallel way in the classical and intuitionistic case, that is, considering a classical model to be a special case of an intuitionistic model, where $w \leq v$ iff $w = v$, we give below the definition for both at the same time. We write $w \nVdash A$ if w does not force A, i.e. it is not the case that $w \Vdash A$.

$w \nVdash \bot$		
$w \Vdash \top$		
$w \Vdash \neg A$	iff	for all w' with $w \leq w'$, we have $w' \nVdash A$
$w \Vdash A \wedge B$	iff	$w \Vdash A$ and $w \Vdash B$
$w \Vdash A \vee B$	iff	$w \Vdash A$ or $w \Vdash B$
$w \Vdash A \supset B$	iff	for all w' with $w \leq w'$, if $w' \Vdash A$ then also $w' \Vdash B$
$w \Vdash \Box A$	iff	for all w' and u with $w \leq w'$ and $w'Ru$, we have $u \Vdash A$
$w \Vdash \Diamond A$	iff	there is a $u \in W$ such that wRu and $u \Vdash A$

It follows that $\Vdash$ also satisfies monotonicity, i.e. if $w \leq v$ and $w \Vdash A$ then $v \Vdash A$. Note that we always have $w \Vdash \neg A$ iff $w \Vdash A \supset \bot$. In the classical case we also have $w \Vdash \neg A$ iff $w \nVdash A$ which implies the de Morgan dualities, in particular, $w \Vdash \Box(\neg A)$ iff $w \Vdash \neg(\Diamond A)$.

We say that a formula A *is valid in a model* $\mathcal{M}$, if for all $w \in W$ we have $w \Vdash A$. Finally, we say a formula is *classically (or intuitionistically) valid*, if it is valid in all classical (resp. intuitionistic) models.

For proving the soundness of our system, we must also define what is the *validity* of a sequent in a model. For this, we adapt here the method of Fitting [Fit15] to the intuitionistic setting. The first step is to put intuitionistic indexed nested sequent in correspondence with intuitionistic models.

Definition 5.1 Let Σ be an indexed nested sequent. We write I_Σ to denote the set of indexes occurring in Σ. We write R_Σ for the accessibility relation induced by Σ, that is, the binary relation $R_\Sigma \subseteq I_\Sigma \times I_\Sigma$ defined as: $wR_\Sigma v$ iff $\Sigma = \Gamma^{w}\{[^{v}\Delta]\}$ for some $\Gamma\{\ \}$ and Δ, i.e. v is the index of a child of w.

Example 5.2 If we consider the sequent Σ obtained in the Example 2.1, we have that $I_\Sigma = \{0, 1, 2, 3, 4, 5\}$ with 0 being the index of the root, so $R_\Sigma = \{(0, 1), (0, 2), (0, 3), (1, 2), (2, 4), (2, 5), (3, 1)\}$

Definition 5.3 Let Σ be an indexed nested sequent and let $\mathcal{M} = \langle W, R, \leq, V\rangle$ be an intuitionistic Kripke model. A *homomorphism* [3] $h \colon \Sigma \to \mathcal{M}$ is a mapping $h \colon I_\Sigma \to W$, such that $wR_\Sigma v$ implies $h(w)Rh(v)$ for all $w, v \in I_\Sigma$.

A preorder relation between homomorphisms can be obtained from the preorder in an intuitionistic model: For $h, h' \colon \Sigma \to \mathcal{M}$ two homomorphisms, we write $h \leq h'$ if $h(w) \leq h'(w)$ in $\mathcal{M}$ for all $w \in I_\Sigma$.

[3] This definition corresponds to the notion of *structural mapping* in [Fit15].

The notion of validity can then be defined by induction on the subsequents of a given sequent. However, the correspondence between indexes in a sequent and worlds in a model brings us to consider the particular class of exhaustive subsequents.

Definition 5.4 Let Σ and Δ be indexed nested sequents, and $w \in I_\Sigma$. We say that $\langle \Delta, w \rangle$ is an *exhaustive subsequent* of Σ if either $\Delta = \Sigma$ and $w = 0$, or $\Sigma = \Gamma\{[^w \Delta]\}$ for some context $\Gamma\{\ \}$.

Note that for a given index v of Σ, there might be more than one Δ such that $\langle \Delta, v \rangle$ is an exhaustive subsequent of Σ, simply because v occurs more than once in Σ. For this reason we will write $\dot{v}$ to denote a particular occurrence of v in Σ and $\Sigma|_{\dot{v}}$ for the subsequent of Σ rooted at the node $\dot{v}$. $\langle \Sigma|_{\dot{v}}, v \rangle$ stands then for a uniquely defined exhaustive subsequent of Σ.

Definition 5.5 Let $h \colon \Sigma \to \mathcal{M}$ be a homomorphism from a sequent Σ to a model $\mathcal{M}$. Let $w \in I_\Sigma$ and let $\langle \Delta, w \rangle$ be an exhaustive subsequent of Σ. From (9) and (10), Δ has one of the following forms:

- $\Delta = B_1^\bullet, \ldots, B_l^\bullet, [^{v_1} \Lambda_1^\bullet], \ldots, [^{v_n} \Lambda_n^\bullet]$. Then we define $\langle h, w \rangle \Vdash_{\mathsf{i}} \Delta$ if $h(w) \nVdash B_i$ for some $i \leq l$ or $\langle h, v_j \rangle \Vdash_{\mathsf{i}} \Lambda_j^\bullet$ for some $j \leq n$
- $\Delta = B_1^\bullet, \ldots, B_l^\bullet, [^{v_1} \Lambda_1^\bullet], \ldots, [^{v_n} \Lambda_n^\bullet], A^\circ$. Then we define $\langle h, w \rangle \Vdash_{\mathsf{i}} \Delta$ if either $h(w) \nVdash B_i$ for some $i \leq l$ or $\langle h, v_j \rangle \Vdash_{\mathsf{i}} \Lambda_j^\bullet$ for some $j \leq n$ or $h(w) \Vdash A$.
- $\Delta = B_1^\bullet, \ldots, B_l^\bullet, [^{v_1} \Lambda_1^\bullet], \ldots, [^{v_n} \Lambda_n^\bullet], [^u \Pi^\circ]$. Then we define $\langle h, w \rangle \Vdash_{\mathsf{i}} \Delta$ if either $h(w) \nVdash B_i$ for some $i \leq l$ or $\langle h, v_j \rangle \Vdash_{\mathsf{i}} \Lambda_j^\bullet$ for some $j \leq n$ or for all homomorphisms $h' \geq h$, we have that $\langle h', u \rangle \Vdash_{\mathsf{i}} \Pi^\circ$.

If, for all $h' \geq h$, $\langle h', w \rangle \Vdash_{\mathsf{i}} \Delta$, we say that $\langle \Delta, w \rangle$ *is intuitionistically valid in $\mathcal{M}$ under h*. Then, a sequent Σ is *valid* in a model $\mathcal{M}$, if $\langle \Sigma, 0 \rangle$ is valid in $\mathcal{M}$ under every $h \colon \Sigma \to \mathcal{M}$.

Informally, an indexed nested sequent is valid if it contains anywhere in the sequent tree a valid output formula or an invalid input formula . The following lemma formalises this idea.

Lemma 5.6 *Let Σ be an indexed nested sequent. Let $\langle \Delta, v \rangle$ be a exhaustive subsequent of Σ. Suppose $\Delta = \Gamma^w\{A\}$ for some context $\Gamma^w\{\ \}$ and some formula A. Let $\mathcal{M}$ be a Kripke model and $h \colon \Sigma \to \mathcal{M}$ a homomorphism.*

- *If $A = A^\circ$ and $h(w) \Vdash A$, then $\langle h, v \rangle \Vdash_{\mathsf{i}} \Delta$.*
- *If $A = A^\bullet$ and $h(w) \nVdash A$, then $\langle h, v \rangle \Vdash_{\mathsf{i}} \Delta$.*

Proof We prove this result by induction on the tree rooted at the considered occurrence of v. In the first case, there are the two following possibilities:

- either $\Delta = A^\circ, \Theta$, then $\langle h, v \rangle \Vdash_{\mathsf{i}} \Delta$ iff $\langle h, v \rangle \Vdash_{\mathsf{i}} \Theta$ or $h(v) \Vdash A$, but as here $v = w$ and we have $h(w) \Vdash A$, it is trivially true;
- or $\Delta = \Theta, [^u \Pi^w\{A^\circ\}]$, and then $\langle h, v \rangle \Vdash_{\mathsf{i}} \Delta$ iff $\langle h, v \rangle \Vdash_{\mathsf{i}} \Theta$ or for all $h' \geq h$, $\langle h', u \rangle \Vdash_{\mathsf{i}} \Pi^w\{A^\circ\}$, which is true by the induction hypothesis.

In the second case, there are the three following possibilities:

– either $\Delta = A^\bullet, \Theta$, then $\langle h, v\rangle \Vdash_{\mathsf{i}} \Delta$ iff $\langle h, v\rangle \Vdash_{\mathsf{i}} \Theta$ or $h(v) \nVdash A$, but as here $v = w$ and we have $h(w) \nVdash A$, it is trivially true;
– or $\Delta = \Theta, [{}^{u}\Pi^{\,w}\{A^\bullet\}]$ where Π contains an output formula, and then $\langle h, v\rangle \Vdash_{\mathsf{i}} \Delta$ iff $\langle h, v\rangle \Vdash_{\mathsf{i}} \Theta$ or for all $h' \geq h$, $\langle h', u\rangle \Vdash_{\mathsf{i}} \Pi^{\,w}\{A^\bullet\}$, which is true by the induction hypothesis;
– or $\Delta = \Theta, [{}^{u}\Lambda^{\,w}\{A^\bullet\}]$ where Λ does not contain any outout formula, and then $\langle h, v\rangle \Vdash_{\mathsf{i}} \Delta$ iff $\langle h, v\rangle \Vdash_{\mathsf{i}} \Theta$ or $\langle h, u\rangle \Vdash_{\mathsf{i}} \Lambda^{\,w}\{A^\bullet\}$, which is true by the induction hypothesis. ☐

We now make explicit the class of model that we are going to consider in order to interpret system $\mathsf{iNIK}+\mathsf{G}$. We are going to consider the notion of graph-consistency introduced by Simpson [Sim94] because we think it helps to glimpse the intricacies of the formalism.

Definition 5.7 Let $\mathcal{M} = \langle W, R, \leq, V\rangle$ be a be an intuitionistic model and let $\langle k, l, m, n\rangle \in \mathbb{N}^4$. We say that $\mathcal{M}$ is a $\mathsf{g}(k, l, m, n)$*-model* if for all $w, u, v \in W$ with $wR^k u$ and $wR^m v$ there is a $z \in W$ such that $uR^l z$ and $vR^n z$.[4] For a set $\mathbb{G}$ of $\mathbb{N}^4$-tuples, we say that $\mathcal{M}$ is a $\mathbb{G}$*-model*, if for all $\langle k, l, m, n\rangle \in \mathbb{G}$ we have that $\mathcal{M}$ is a $\mathsf{g}(k, l, m, n)$-model.

Definition 5.8 A intuitionistic model $\mathcal{M}$ is called *graph-consistent* if for any sequent Γ, given any homomorphism $h\colon \Gamma \mapsto \mathcal{M}$, any $w \in I_\Gamma$, and any $w' \geq h(w)$, there exists $h' \geq h$ such that $h'(w) = w'$.

We finally prove that any theorem of $\mathsf{iNIK}+\mathsf{G}$ is valid in every graph-consistent $\mathbb{G}$-model by showing that each rule of $\mathsf{iNIK}+\mathsf{G}$ is sound when interpreted in these models.

Lemma 5.9 *Let $\mathbb{G} \subseteq \mathbb{N}^4$, and let* $\mathsf{r}\,\dfrac{\Sigma_1 \quad \cdots \quad \Sigma_n}{\Sigma}$ *be an instance of an inference rule in $\mathsf{iNIK}+\mathsf{G}$ for $n = 0, 1, 2$. If all of $\Sigma_1, \ldots, \Sigma_n$ are valid in every graph-consistent $\mathbb{G}$-model, then so is Σ.*

Proof First, assume that r is $\mathsf{g_{k,l,m,n}}\,\dfrac{\Phi}{\Psi}$, for some $\langle k, l, m, n\rangle \in \mathbb{G}$ such that $k, l, m, n > 0$ (similar proof when one parameter is 0). By way of contradiction, suppose that Φ is valid in every graph-consistent $\mathbb{G}$-model and that there is a $\mathbb{G}$-model $\mathcal{M} = \langle W, R, \leq, V\rangle$, a homomorphism $h\colon \Psi \to \mathcal{M}$ such that $\langle \Psi, 0\rangle$ is not valid in $\mathcal{M}$ under h. Recall that Ψ is of form

$$\Psi = \Gamma^{\,u_0}\{[{}^{u_1}\Delta_1, \ldots [{}^{u_k}\Delta_k] \ldots], [{}^{w_1}\Sigma_1, \ldots [{}^{w_m}\Sigma_m] \ldots]\}$$

Therefore, there exist $\mathtt{u}_0, \mathtt{u}_k, \mathtt{w}_m$ in W such that $\mathtt{u}_0 = h(u_0)$, $\mathtt{u}_k = h(u_k)$, $\mathtt{w}_m = h(w_m)$, and $\mathtt{u}_0 R^k \mathtt{u}_k$, and $\mathtt{u}_0 R^m \mathtt{w}_m$ (Definitions 5.1 and 5.3). Hence, as $\mathcal{M}$ is in particular a $\mathsf{g}(k, l, m, n)$-model, there exists $\mathtt{y} \in W$ with $\mathtt{u}_k R^l \mathtt{y}$ and $\mathtt{w}_m R^n \mathtt{y}$

[4] We define the composition of two relations R, S on a set W as usual: $R \circ S = \{(w, v) \mid \exists u.\ (wRu \wedge uSv)\}$. R^n stands for R composed n times with itself.

(Definition 5.7). Namely, there are worlds $\mathtt{v}_1, \ldots, \mathtt{v}_l, \mathtt{x}_1, \ldots, \mathtt{x}_n$ in W such that $\mathtt{u}_k R \mathtt{v}_1 \ldots \mathtt{v}_{l-1} R \mathtt{v}_l$, $\mathtt{w}_m R \mathtt{x}_1 \ldots \mathtt{x}_{n-1} R \mathtt{x}_n$, and $\mathtt{v}_l = \mathtt{y} = \mathtt{x}_n$. By noting that

$$\Phi = \Gamma^{u_0}\{[^{u_1}\Delta_1, ...[^{u_k}\Delta_k, [^{v_1} ...[^{v_l}\ ]...]]...], [^{w_1}\Sigma_1, ...[^{w_m}\Sigma_m, [^{x_1} ...[^{x_n}\ ]...]]...]\}$$

we can define a homomorphism $h' \colon \Phi \to \mathcal{M}$ with $h'(z) = h(z)$ for all $z \in I_\Psi$, $h'(v_i) = \mathtt{v}_i$ for $1 \leq i \leq l$ and $h'(x_j) = \mathtt{x}_j$ for $1 \leq j \leq n$.

We are now going to show that for every $h \colon \Psi \to \mathcal{M}$, and every occurrence $\dot{z}$ of an index $z \in I_\Psi$, we have $\langle h, z\rangle \Vdash_{\mathsf{i}} \Psi|_{\dot{z}}$ iff $\langle h', z\rangle \Vdash_{\mathsf{i}} \Phi|_{\dot{z}}$. We proceed by induction on the height of the tree rooted at $\dot{z}$.

1. The node of $\dot{z}$ is a leaf node of Ψ, and $z \neq u_k$ and $z \neq w_m$. Then we have $\Psi|_{\dot{z}} = \Phi|_{\dot{z}}$ and the claim holds trivially.
2. The node of $\dot{z}$ is an inner node of Ψ, and $z \neq u_k$ and $z \neq w_m$. By the induction hypothesis, for every $t \in I_\Psi$ with $zR_\Psi t$, every occurrence $\dot{t}$ of t in $\Psi|_{\dot{z}}$, and every $h \colon \Psi \to \mathcal{M}$, $\langle h, t\rangle \Vdash_{\mathsf{i}} \Psi|_{\dot{t}}$ iff $\langle h', t\rangle \Vdash_{\mathsf{i}} \Phi|_{\dot{t}}$. The statement follows then by unravelling the definition of $\Vdash_{\mathsf{i}}$ (Definition 5.5).
3. $z = u_k$. For any occurrence $\dot{z}$ in the context $\Gamma^{z_0}\{\ \}$, the proof is similar to one of the previous cases. Otherwise, we know that $\Psi|_{\dot{z}} = \Delta_k$ and $\Phi|_{\dot{z}} = \Delta_k, [^{v_1}...[^{v_l}\ ]...]$. Furthermore, for all $i \leq l$ and $h'' \geq h$ we have $\langle h'', v_i\rangle \nVdash_{\mathsf{i}} [^{v_{i+1}}...[^{v_l}\ ]...]$, and therefore $\langle h, z\rangle \Vdash_{\mathsf{i}} \Psi|_{\dot{z}}$ iff $\langle h', z\rangle \Vdash_{\mathsf{i}} \Phi|_{\dot{z}}$.
4. $v = w_m$. This case is similar to the previous one.

Since we assumed that $\langle \Psi, 0\rangle$ is not valid in $\mathcal{M}$ under h, we can conclude that $\langle \Phi, 0\rangle$ is not valid in $\mathcal{M}$ under h', contradicting the validity of Φ.

The proof for bc, tp, and the other cases of $\mathsf{g_{k,l,m,n}}$ when one of the parameters is 0, is similar. For the logical rules, we will consider in details the case for $\Box^\circ$, the others are similar (the cases for $\supset^\circ$ and $\neg^\circ$ also make use of the graph-consistency property).

Suppose that $\Phi = \Gamma^{w}\{[^{v}A^\circ]\}$ is valid in every graph-consistent $\mathbb{G}$-model. For $\Psi = \Gamma^{w}\{\Box A^\circ\}$, suppose that there exists a graph-consistent $\mathbb{G}$-model $\mathcal{M} = \langle W, R, \leq, V\rangle$ and a homomorphism $h \colon \Psi \mapsto \mathcal{M}$ such that $\langle \Psi, 0\rangle$ is not valid in $\mathcal{M}$ under h. Therefore, there exists $h' \geq h$ such that $\langle h', 0\rangle \nVdash_{\mathsf{i}} \Psi$, in particular by Lemma 5.6, $h'(w) \nVdash \Box A$. So there exists $\mathtt{w}$ and $\mathtt{v}$ such that $\mathtt{w} R \mathtt{v}$, $h'(w) \leq \mathtt{w}$ and $\mathtt{v} \nVdash A$. As $\mathcal{M}$ is graph-consistent, there exists h'' such that $\mathtt{w} = h''(w)$. Thus, we can extend h'' by setting $h''(v) = \mathtt{v}$ to obtain a homomorphism $h'' \colon \Phi \mapsto \mathcal{M}$, indeed Φ and Ψ have the same set of indexes related by the same underlying structure, but for the fresh index v that does not appear in Ψ. Finally, as $h''(v) \nVdash A$, we have by Lemma 5.6 that $\langle \Phi, 0\rangle$ is not valid in $\mathcal{M}$ under h'' which contradicts the assumption of validity of Φ. $\square$

We recall that the *height* of a derivation tree π, denoted by $\mathsf{ht}(\pi)$, is the length of the longest path in the tree from its root to one of its leaves.

Theorem 5.10 *Let $\mathbb{G} \subseteq \mathbb{N}^4$ and G be the corresponding set of rules. If a sequent Σ is provable in $\mathsf{iNIK} + \mathsf{G}$ then it is valid in every graph-consistent intuitionistic $\mathbb{G}$-model.*

Proof By induction on the height of the derivation, using Lemma 5.9. $\square$

The soundness result in [Fit15] can of course be obtained as a corollary of this theorem, as this proof method extended Fitting's technique to the intuitionistic framework.

Corollary 5.11 *Let $\mathbb{G} \subseteq \mathbb{N}^4$ and G be the corresponding set of rules. If a sequent Σ is provable in $\mathsf{iNK}_2 + \mathsf{G}$ then it is valid in every classical $\mathbb{G}$-model.*

6 Discussion

In the classical case, for a given $\mathbb{G} \subseteq \mathbb{N}^4$, and G be the corresponding set of rules, the logic given by $\mathsf{iNK}_2 + \mathsf{G}$ (equivalent to Fitting's system [Fit15]) corresponds exactly to the logic axiomatised by the extension of the Hilbert system K with the corresponding Scott-Lemmon axioms G. This follows from what we presented in the two previous sections. Theorem 4.3 ensures that every theorem of $\mathsf{K} + \mathsf{G}$ is a theorem of $\mathsf{iNK}_2 + \mathsf{G}$ (as a corollary of cut-elimination in $\mathsf{iNK}_2 + \mathsf{G}$). Then, Theorem 5.11 shows that every theorem of $\mathsf{iNK}_2 + \mathsf{G}$ is valid in any classical $\mathbb{G}$-model. In the end, the cornerstone that allows one to conclude that $\mathsf{iNK}_2 + \mathsf{G}$ is indeed sound and complete wrt. $\mathsf{K} + \mathsf{G}$, is that the Hilbert system $\mathsf{K} + \mathsf{G}$ actually completely axiomatises classical $\mathbb{G}$-models.

Theorem 6.1 (Lemmon and Scott [LS77]) *Let $\mathbb{G} \subseteq \mathbb{N}^4$. A formula is derivable in $\mathsf{K} + \mathsf{G}$, iff it is valid in all classical $\mathbb{G}$-models.*

This means that in the classical case, we have a complete triangle between Kripke models, Hilbert axiomatisation and nested sequents systems via Theorems 4.2, 5.11 and 6.1.

In the intuitionistic case, the correspondence theory is much more tedious, and a lot of questions are still open. We do have Theorems 5.10 giving that every theorem of $\mathsf{IK} + \mathsf{G}$ is a theorem of $\mathsf{iNIK} + \mathsf{G}$, and Theorem 4.2 giving that every theorem of $\mathsf{iNIK} + \mathsf{G}$ is valid in graph-consistent $\mathbb{G}$-models, but there is no proper equivalent to Theorem 6.1 to "link" the two theorems into an actual soundness and completeness result for $\mathsf{iNIK} + \mathsf{G}$. As we have seen in Section 4, the first inclusion is strict, since the formula in (11) is provable in $\mathsf{iNIK} + \mathsf{G}$, but not in $\mathsf{IK} + \mathsf{G}$. However, the strictness of the second inclusion is open. The question is: Is there a certain set $\mathbb{G} \subseteq \mathbb{N}^4$ such that there exists a formula that is valid in every directed graph-consistent $\mathbb{G}$-models, but that is not a theorem of $\mathsf{iNIK} + \mathsf{G}$, for G the set of rules corresponding to $\mathbb{G}$?

On the other hand, Theorems 6.2.1 and 8.1.4 of [Sim94] entail a parallel result to Theorem 6.1 for a restricted family of the intuitionistic Scott-Lemmon axioms, those for which $l = 1$ and $n = 0$ (or equivalently $l = 0$ and $n = 1$), that is, of the form: $(\Diamond^k \Box A \supset \Box^m A) \wedge (\Diamond^k A \supset \Box^m \Diamond A)$. Therefore, in this restricted case, the inclusions collapse too. The reason why this result holds seems to be that in a derivation of a theorem of such a logic, the steps referring to non-tree graphs can be eliminated via appealing to the closure of the accessibility relation (see [Sim94]). This is very similar to what happens when going from indexed to pure nested sequents calculi, and suggests that a pure nested sequent calculus could be provided for these logics in the intuitionistic case too. Indeed, these axioms are exactly the intuitionistic variants of the ones called *path axioms* in [GPT11], for which a pure nested sequent calculus is given; but for the general case, this paper only provides a display calculus.

To conclude, we can say that for intuitionistic modal logics the accurate definition might actually come from structural proof-theoretical studies rather than Hilbert axiomatisations or semantical considerations. For Simpson [Sim94] there are two different (but equivalent) ways to define intuitionistic modal logics, either the natural deduction systems he proposes, or the extension of the standard translation for intuitionistic modal logics into first-order intuitionistic logic. Equivalence between the natural deduction systems and the Hilbert axiomatisations, or direct interpretation of the natural deduction systems in intuitionistic (birelational) structures are just side-results. He therefore sees their failure for the majority of logics not as a problem, but rather as another justification of the validity of the proof-theoretic approach.

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